

# Comment on “Reply to Comment on Time-dependent Quasi-Hermitian Hamiltonians and the Unitary Quantum Evolution”

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## Abstract

I point out that if one defines the operator  $U_R(t)$  as done by M. Znojil in his reply [arXiv:0711.0514v1] to my comment [arXiv:0711.0137v1] and also accepts the validity of the defining relation of  $U_R(t)$  as given in his paper [arXiv:0710.5653v1], one finds that the time-evolution of the associated quantum system is not governed by the Schrödinger equation for the Hamiltonian operator  $H$  but an operator  $H'$  which differs from  $H$  if the metric operator is time-dependent. In the latter case this effective Hamiltonian  $H'$  is not observable. This is consistent with the conclusions of my paper [Phys. Lett. B **650**, 208 (2007), arXiv:0706.1872v2] which allow for unitary time-evolution generated by unobservable Hamiltonians.

In [1] M. Znojil explains that my comment [2] on his paper [3] is not relevant, because it relies on the relations

$$i\hbar\partial_t U_R(t) = H(t)U_R(t), \quad U_R(0) = I, \quad (1)$$

which he identifies as an “*incorrect*” assumption. To clarify the matter, recall that  $U_R(t)$  was initially introduced in Eq. (14) of [3] which reads

$$|\Phi(t)\rangle = U_R(t)|\Phi(0)\rangle, \quad (2)$$

where  $|\Phi(t)\rangle$  is an arbitrary evolving state vector.

Clearly (1) follows from (2), if one postulates the standard Schrödinger time-evolution defined by the Hamiltonian operator  $H$ , namely

$$i\hbar\partial_t |\Phi(t)\rangle = H(t)|\Phi(t)\rangle. \quad (3)$$

This shows that in M. Znojil's formulation the time-evolution is not defined by the Hamiltonian  $H$ .

If one insists on defining  $U_R(t)$  by the relation

$$U_R(t) = \omega(t)^{-1} u(t) \omega(0), \quad (4)$$

as done in [1], then (2) together with  $i\hbar\partial_t u(t) = h(t)u(t)$  and  $h(t) = \omega(t)H(t)\omega(t)^{-1}$  that are respectively listed as Eqs. (16) and (19) of [3] yield

$$i\hbar\partial_t|\Phi(t)\rangle = [H(t) - i\hbar\omega(t)^{-1}\partial_t\omega(t)]|\Phi(t)\rangle. \quad (5)$$

Therefore, in M. Znojil's scheme, the evolving state vector is determined not by  $H(t)$  but by another operator namely

$$H'(t) := H(t) - i\hbar\omega(t)^{-1}\partial_t\omega(t). \quad (6)$$

It is easy to see that  $H'$  coincides with  $H$  if and only if the metric operator is time-independent.

An important consequence of allowing a time-dependent metric operator  $\Theta$  in M. Znojil's scheme is the non- $\Theta$ -pseudo-Hermiticity of  $H'(t)$ . This in turn means that  $H'(t)$  is not an observable. The situation becomes clear if we recall the main result of [4] namely:

**Theorem:** *If the metric operator is time-dependent, then the observability of the Hamiltonian (the generator of the Schrödinger time-evolution) is inconsistent with the unitarity of time evolution.*

In M. Znojil's scheme the generator of the Schrödinger time-evolution, i.e.,  $H'(t)$ , is not an observable. Therefore, the unitarity of the Schrödinger time-evolution generated by  $H'(t)$  does not conflict with the above theorem.

Next, I would like to make two comments.

1. One can rewrite (5) in the form

$$i\hbar D_t|\Phi(t)\rangle = H(t)|\Phi(t)\rangle, \quad D_t := \partial_t + \omega(t)^{-1}\partial_t\omega(t), \quad (7)$$

and identify  $D_t$  with a covariant time-derivative. Such an approach has actually been pursued long ago in the context of quantum cosmology [5].

2. A more straightforward formulation of the approach of [3] which does not involve any specific notation and is mathematically unambiguous is as follows. Suppose that  $\mathcal{H}$  is a reference Hilbert space with inner product  $\langle\cdot|\cdot\rangle$ ,  $\Theta : \mathcal{H} \rightarrow \mathcal{H}$  is a possibly time-dependent (positive) metric operator,  $\mathcal{H}_{\text{phys}}$  is the Hilbert space with the same vector space structure as  $\mathcal{H}$  and the inner product

$$\prec\cdot,\cdot\succ := \langle\cdot|\Theta\cdot\rangle, \quad (8)$$

and  $H : \mathcal{H} \rightarrow \mathcal{H}$  be a possibly time-dependent  $\Theta$ -pseudo-Hermitian operator. Suppose that the state vectors  $\Phi(t)$  evolve according to

$$\Phi(t) = U(t)\Phi(0), \quad (9)$$

where  $U(t) : \mathcal{H} \rightarrow \mathcal{H}$  is a linear densely-defined invertible operator satisfying  $U(0) = I$  and  $I$  is the identity operator. The unitarity of this evolution in the Hilbert space  $\mathcal{H}_{\text{phys}}$ , i.e.,

$$\prec \Phi(t), \Phi(t) \succ = \prec \Phi(0), \Phi(0) \succ, \quad (10)$$

is equivalent to

$$U(t)^\dagger \Theta(t) U(t) = \Theta(0), \quad (11)$$

where  $U(t)^\dagger$  is the unique operator satisfying  $\langle \xi | U(t)^\dagger \zeta \rangle = \langle U(t) \xi | \zeta \rangle$  for all  $\xi, \zeta \in \mathcal{H}$ . So far  $H(t)$  does not play any role in this scheme. Let  $H' : \mathcal{H} \rightarrow \mathcal{H}$  be defined by

$$H'(t) := i\hbar[\partial_t U(t)]U(t)^{-1}. \quad (12)$$

Then  $\Phi(t)$  is a solution of the Schrödinger equation for a Hamiltonian operator  $H'(t)$ ,

$$i\hbar\partial_t\Phi(t) = H'(t)\Phi(t). \quad (13)$$

Differentiating (11) and using (12) in the resulting equation, we also find

$$H'^\dagger = \Theta(t)H'(t)\Theta(t)^{-1} + i\hbar[\partial_t\Theta(t)]\Theta(t)^{-1}. \quad (14)$$

The Hamiltonian  $H'$  is  $\Theta$ -pseudo-Hermitian, i.e., an observable, if and only if  $\Theta$  is constant. This argument is independent of how one relates  $H'(t)$  to  $H(t)$ , and it is in complete agreement with the above theorem.

In conclusion, M. Znojil formulation of dynamics [3] allows for a unitary time-evolution with respect to a time-dependent inner product, but this dynamics is generated by an operator that is not an observable. If one uses the standard notion of the “Hamiltonian” of a quantum system, namely as the generator of the Schrödinger time-evolution, then the Hamiltonian becomes unobservable. This is actually a direct implication of the above theorem. What has been done in [3] is to use the term “Hamiltonian” for a different purpose. Such a non-standard use of terminology is at the root of M. Znojil’s disagreement with the results of [4].

## References

- [1] M. Znojil, ‘Reply to Comment on Time-dependent quasi-Hermitian Hamiltonians and the Unitary Quantum Evolution,’ arXiv:0711.0514v1.
- [2] A. Mostafazadeh, ‘Comment on Time-dependent Quasi-Hermitian Hamiltonians and the Unitary Quantum Evolution,’ arXiv:0711.0137v1.
- [3] M. Znojil, ‘Time-dependent Quasi-Hermitian Hamiltonians and the Unitary Quantum Evolution,’ arXiv:0710.5653v1.
- [4] A. Mostafazadeh, Phys. Lett. B **650**, 208 (2007), arXiv:0706.1872v2.
- [5] C. Isham, ‘Canonical Quantum Gravity and the Problem of Time,’ in *Integrable Systems, Quantum Groups, and Quantum Field Theories*, NATO ASI Series, editors: L. A. Ibort and M. A. Rodriguez, Kluwer Academic Publishers, 1993, pp. 157-287.